Unbiased shifts of Brownian motion

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Abstract

Let $B = (B_t)_{t\in\mathbb{R}}$ be a two-sided standard Brownian motion. An *unbiased shift* of B is a random time T, which is a measurable function of B, such that $(B_{T+t} - B_T)_{t\in\mathbb{R}}$ is a Brownian motion independent of B_T. We characterise unbiased shifts in terms of allocation rules balancing additive functionals of B. For any probability distribution ν on $\mathbb{R}$ we construct a stopping time $T \geq 0$ with the above properties such that B_T has distribution ν. In particular, we show that if we travel in time according to the clock of local time we always see a two-sided Brownian motion. A crucial ingredient of our approach is a new theorem on the existence of allocation rules balancing jointly stationary diffuse random measures on $\mathbb{R}$. We also study moment and minimality properties of unbiased shifts.

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1 Introduction and main results

Let $B = (B_t)_{t\in\mathbb{R}}$ be a two-sided standard Brownian motion in $\mathbb{R}$ having $B_0 = 0$. If $T \geq 0$ is a stopping time with respect to the filtration $(\sigma\{B_s : s \leq t\})_{t\geq 0}$, then the shifted process $(B_{T+t} - B_T)_{t\geq 0}$ is a one-sided Brownian motion independent of B_T. However, the two-sided shifted process $(B_{T+t} - B_T)_{t\in\mathbb{R}}$ need not be a two-sided Brownian motion. Moreover, the example of a fixed time shows that even if it is, it need not be independent of B_T. We call a random time T an *unbiased shift* (of a two-sided Brownian motion) if T is a measurable function of B and $(B_{T+t} - B_T)_{t\in\mathbb{R}}$ is a two-sided Brownian motion, independent of B_T. We say that a random time T *embeds* a given probability measure ν on $\mathbb{R}$, often called the *target distribution*, if B_T has distribution ν.

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In this paper we discuss several examples of nonnegative unbiased shifts that are stopping times. However, we wish to stress that nonnegative unbiased shifts are not assumed to have the stopping time property, see for instance Example 5.8. The paper has three main aims. The first aim is to characterise all unbiased shifts that embed a given distribution ν. The second aim is to construct such unbiased shifts. In particular, we solve the Skorokhod embedding problem for unbiased shifts: given any target distribution we find an unbiased shift which embeds this target distribution (and is also a stopping time). The third and final aim is to discuss some properties of unbiased shifts. In particular, we discuss optimality of our solution of the Skorokhod embedding problem for unbiased shifts.

The case when the embedded distribution is concentrated at zero is of special interest. Let ℓ^0 be the local time at zero. Its right-continuous (generalised) inverse is defined by

$$T_r := \begin{cases} \sup\{t \geq 0 : \ell^0[0,t] = r\}, & r \geq 0, \\ \sup\{t < 0 : \ell^0[t,0] = -r\}, & r < 0. \end{cases} \tag{1.1}$$

Note that $\mathbb{P}_0\{T_0 = 0\} = 1$ and $\mathbb{P}_0\{T_r = 0\} = 0$ if $r \neq 0$. We prove the following theorem.

Theorem 1.1. *Let $r \in \mathbb{R}$. Then T_r is an unbiased shift embedding δ_0.*

This result formalises the intuitive idea that two-sided Brownian motion looks globally the same from all its (appropriately chosen) zeros, thus resolving an issue raised by Mandelbrot in [15, p. 207, p. 385] and reinforced in [11, 24]. Another way of thinking about this result is that if we travel in time according to the clock of local time we always see a two-sided Brownian motion. This is analogous to a well-known property of the two-sided stationary Poisson process with an extra point at the origin: the lengths of the intervals between points are i.i.d. (exponential) and therefore shifting the origin to the nth point on the right (or on the left) gives us back a two-sided Poisson process with an extra point at the origin. In fact, much of the work behind the present paper was inspired by recent developments for spatial point processes and random measures [9, 8, 14]. In the terminology of [14], Brownian motion is *mass-stationary* with respect to local time, see Section 3.

Theorem 1.1 is relatively elementary. To state the further main results of this paper, we need to introduce some notation and terminology. It is convenient to define B as the identity on the canonical probability space $(\Omega, \mathcal{A}, \mathbb{P}_0)$, where Ω is the set of all continuous functions $\omega\colon \mathbb{R} \to \mathbb{R}$, $\mathcal{A}$ is the Kolmogorov product σ-algebra, and $\mathbb{P}_0$ is the distribution of B. Define $\mathbb{P}_x := \mathbb{P}_0\{B + x \in \cdot\}$, $x \in \mathbb{R}$, and the σ-finite and stationary measure (see Section 2)

$$\mathbb{P} := \int \mathbb{P}_x \, dx. \tag{1.2}$$

Expectations (resp. integrals) with respect to $\mathbb{P}_x$ and $\mathbb{P}$ are denoted by $\mathbb{E}_x$ and $\mathbb{E}$, respectively. For any $t \in \mathbb{R}$ the shift $\theta_t\colon \Omega \to \Omega$ is defined by

$$(\theta_t \omega)_s := \omega_{t+s}.$$

An *allocation rule* [9, 14] is a measurable mapping $\tau\colon \Omega \times \mathbb{R} \to \mathbb{R}$ that is *equivariant* in the sense that

$$\tau(\theta_t \omega, s - t) = \tau(\omega, s) - t, \quad s, t \in \mathbb{R}, \ \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{1.3}$$

A *random measure* ξ on $\mathbb{R}$ is a kernel from Ω to $\mathbb{R}$ such that $\xi(\omega, C) < \infty$ for $\mathbb{P}$-a.e. ω and all compact $C \subset \mathbb{R}$. If ξ and η are random measures, and τ is an allocation rule such that the image measure of ξ under τ is η, that is,

$$\int \mathbf{1}\{\tau(s) \in \cdot\}\, \xi(ds) = \eta \quad \mathbb{P}\text{-a.e.}, \tag{1.4}$$

then we say that τ *balances* ξ and η. If τ balances ξ and η and σ is an allocation rule that balances η and another random measure ζ, then the allocation rule $s \mapsto \sigma(\tau(s))$ balances η and ζ.

Let ℓ^x be the random measure associated with the *local time* of B at $x \in \mathbb{R}$ (under $\mathbb{P}_0$), see also Section 2. If ν is a probability measure on $\mathbb{R}$, then (2.9) below implies that

$$\ell^\nu(\omega, \cdot) := \int \ell^x(\omega, \cdot)\, \nu(dx), \quad \omega \in \Omega,$$

defines a random measure ℓ^ν. This is an *additive functional* of Brownian motion, that is, it has the invariance property (2.2), see Section 2 for more details. In fact, any diffuse additive functional must be of this form with a σ-finite *Revuz measure* ν, see e.g. [12, Chapter 22]. For any random time T we define an allocation rule τ_T by

$$\tau_T(t) := T \circ \theta_t + t, \quad t \in \mathbb{R}. \tag{1.5}$$

Since $T = \tau_T(0)$, there is a one-to-one correspondence between T and τ_T.

Our key characterisation theorem is based on a result in [14], which will be recalled as Theorem 2.1 below.

Theorem 1.2. *Let T be a random time and ν be a probability measure on $\mathbb{R}$. Then T is an unbiased shift embedding ν if and only if τ_T balances ℓ^0 and ℓ^ν.*

For any probability measure ν on $\mathbb{R}$ we denote by $\mathbb{P}_\nu := \int \mathbb{P}_x\, \nu(dx)$ the distribution of a two-sided Brownian motion with a random starting value B_0 with law ν. We show in Section 3 that all these distributions coincide on the invariant σ-algebra. A general result in [23] (see also [12, Theorem 10.28]) then implies that there is a random time T (possibly defined on an extension of $(\Omega, \mathcal{A}, \mathbb{P}_0)$) such that $\theta_T B$ has distribution $\mathbb{P}_\nu$ (under $\mathbb{P}_0$). The next two theorems yield a much stronger result. They show that T can be chosen as a *factor* of B, that is, as a measurable function of B, see [9] for a similar result for Poisson processes. Moreover, this factor is explicitly known. The proof is based on Theorem 1.2 and on a general result on the existence of allocation rules balancing jointly stationary orthogonal diffuse random measures on $\mathbb{R}$ with equal conditional intensities, see Theorem 5.1.

Theorem 1.3. *Let ν be a probability measure on $\mathbb{R}$ with $\nu\{0\} = 0$. Then the stopping time*

$$T := \inf\left\{t > 0\colon \ell^0[0,t] = \ell^\nu[0,t]\right\} \tag{1.6}$$

embeds ν and is an unbiased shift.

The stopping time (1.6) was introduced in [3] as a solution of the *Skorokhod embedding problem*. This problem requires finding a stopping time $T \geq 0$ embedding a given distribution ν, see [19] for a survey. It has apparently not been noticed before that (1.6) is an unbiased shift. If ν is of the form $\nu\{0\}\delta_0 + (1-\nu\{0\})\mu$ where $\mu\{0\} = 0$ and $\nu\{0\} > 0$, then Theorem 1.3 does not apply. In fact, if $\nu\{0\} < 1$ then (1.6) is an unbiased shift embedding μ. Still we can use Theorem 1.3 to construct unbiased shifts without any assumptions on ν:

Theorem 1.4. *Let ν be a probability measure on $\mathbb{R}$. Then there exists a nonnegative stopping time that is an unbiased shift embedding ν.*

In Theorem 1.1 we have $\mathbb{P}_0\{B_{T_0} = 0, T_0 = 0\} = 1$ and $\mathbb{P}_0\{B_{T_r} = 0, T_r \neq 0\} = 1$ if $r \neq 0$. It is interesting to note that unbiased shifts T (even if they are not stopping times) are almost surely non-zero as long as the condition $\mathbb{P}_0\{B_T = 0\} < 1$ is fulfilled:

Theorem 1.5. *Let ν be a probability measure on $\mathbb{R}$ such that $\nu\{0\} < 1$. Then any unbiased shift T embedding ν satisfies*

$$\mathbb{P}_0\{T = 0\} = 0. \tag{1.7}$$

In contrast to the previous theorem, if T is an unbiased shift with $\mathbb{P}_0\{B_T = 0\} = 1$, then the probability $\mathbb{P}_0\{T = 0\}$ may take any value:

Theorem 1.6. *For any $p \in [0, 1]$ there is an unbiased shift $T \geq 0$ embedding δ_0 and such that $\mathbb{P}_0\{T = 0\} = p$.*

A solution T of the Skorokhod embedding problem is usually required to have good moment properties, but some restrictions apply. For instance, if the target distribution ν is not centered, by [17, Theorem 2.50], we must have $\mathbb{E}_0\sqrt{T} = \infty$. If the embedding stopping time is also an unbiased shift the situation is worse, even when ν is centred.

Theorem 1.7. *Suppose ν is a target distribution with $\nu\{0\} = 0$, and $\int |x|\, \nu(dx) < \infty$, and the stopping time $T \geq 0$ is an unbiased shift embedding ν. Then*

$$\mathbb{E}_0 T^{1/4} = \infty.$$

Under the same assumptions on ν the unbiased shift constructed in (1.6) *satisfies*

$$\mathbb{E}_0 T^{\beta} < \infty \text{ for all } \beta < 1/4.$$

Dropping the stopping time assumption we show in Theorem 8.4 that $\mathbb{E}_0\sqrt{|T|} = \infty$ for any unbiased shift T embedding a target distribution ν with $\nu\{0\} < 1$. If the target distribution is concentrated at zero and T is nonnegative but not identically zero, we show in Theorem 8.5 that $\mathbb{E}_0 T = \infty$. Nonnegativity is important in this result, Example 8.6 provides an unbiased shift with $\mathbb{P}_0\{T \neq 0\} = 1$ that has exponential moments.

Theorem 7.5 further shows that, in addition to the nearly optimal moment properties stated above, the stopping times T defined in (1.6) are also *minimal* in a sense analogous to the definition in [18] (see also [4], or [19] for a survey). This means that if $S \geq 0$ is another unbiased shift embedding ν such that $\mathbb{P}_0\{S \leq T\} = 1$, then $\mathbb{P}_0\{S = T\} = 1$.

Our discussion of minimality is based on a notion of stability of allocation rules, which is similar to the one studied in [8].

The structure of the paper is as follows. Section 2 provides some background on Palm measures and local time. Section 3 presents a general result on mass-stationarity for diffuse random measures on the line, implying Theorem 1.1. Section 4 contains the proof of a more general version of Theorem 1.2. Section 5 presents the key general result on balancing diffuse jointly stationary random measures, Theorem 5.1, implying Theorem 1.3. Section 6 contains the proofs of Theorems 1.4, 1.5 and 1.6. In Sections 7 and 8 we discuss minimality and moment properties of unbiased shifts, including the proof of Theorem 1.7. Section 9 concludes with some remarks.

2 Preliminaries on Palm measures and local times

Recall the definition (1.2) of the measure $\mathbb{P}$ on $(\Omega, \mathcal{A})$. Since $\mathbb{P}\{B_0 \in C\} < \infty$ for any compact $C \subset \mathbb{R}$, $\mathbb{P}$ is σ-finite. We also note the invariance property

$$\mathbb{P} = \mathbb{P} \circ \theta_s, \quad s \in \mathbb{R}. \tag{2.1}$$

The proof of (2.1) is only based on the stationary increments of B, see [25]. Corollary 3.3 provides an alternative definition of $\mathbb{P}$. A random measure ξ is called *invariant* if

$$\xi(\theta_t\omega, C - t) = \xi(\omega, C), \quad C \in \mathcal{B}, t \in \mathbb{R}, \mathbb{P}\text{-a.s.}, \tag{2.2}$$

where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. In this case the *Palm measure* $\mathbb{Q}_\xi$ of ξ (with respect to $\mathbb{P}$) is defined by

$$\mathbb{Q}_\xi(A) := \mathbb{E} \int \mathbf{1}_{[0,1]}(s)\mathbf{1}_A(\theta_s B)\, \xi(ds), \quad A \in \mathcal{A}.$$

This is a σ-finite measure on $(\Omega, \mathcal{A})$. If the *intensity* $\mathbb{Q}_\xi(\Omega)$ of ξ is positive and finite, $\mathbb{Q}_\xi$ can be normalised to yield the Palm probability measure of ξ. Even though $\mathbb{Q}_\xi$ is generally not a probability measure, we denote integration with respect to $\mathbb{Q}_\xi$ by $\mathbb{E}_{\mathbb{Q}_\xi}$. The invariance property (2.2) implies the *refined Campbell theorem*

$$\mathbb{E} \int f(\theta_s B, s)\, \xi(ds) = \mathbb{E}_{\mathbb{Q}_\xi} \int f(B, s)\, ds, \tag{2.3}$$

for any measurable $f \colon \Omega \times \mathbb{R} \to [0, \infty)$. We recall the following result from [14].

Theorem 2.1. *Consider two invariant random measures ξ and η on $\mathbb{R}$ and an allocation rule τ. Then τ balances ξ and η if and only if*

$$\mathbb{Q}_\xi\{\theta_\tau B \in \cdot\} = \mathbb{Q}_\eta,$$

where $\theta_\tau \colon \Omega \to \Omega$ is defined by $\theta_\tau(\omega) := \theta_{\tau(\omega,0)}\omega$, $\omega \in \Omega$.

Recall that ℓ^x is the random measure associated with the *local time* of B at $x \in \mathbb{R}$ (under $\mathbb{P}_0$). This means that

$$\int f(B_s, s)\, ds = \iint f(x, s)\, \ell^x(ds)\, dx \quad \mathbb{P}_0\text{-a.s.} \tag{2.4}$$

for all measurable $f\colon \mathbb{R}^2 \to [0,\infty)$. We assume that ℓ^0 is $\mathbb{P}_x$-a.e. diffuse for any $x \in \mathbb{R}$ and that

$$\ell^0(\theta_t\omega, C-t) = \ell^0(\omega, C), \quad C \in \mathcal{B},\, t \in \mathbb{R},\, \mathbb{P}_x\text{-a.s.},\, x \in \mathbb{R}. \tag{2.5}$$

The global construction in [20] guarantees the existence of a version of ℓ^0 with these properties, see also [12, Proposition 22.12] or [17, Theorem 6.43]. This construction implies, moreover, that it is no restriction of generality to assume that

$$\ell^y(\omega,\cdot) = \ell^0(\omega - y,\cdot), \quad \omega \in \Omega,\, y \in \mathbb{R}, \tag{2.6}$$

and

$$\int \mathbf{1}\{B_t \neq x\}\, \ell^x(dt) \equiv 0, \quad x \in \mathbb{R}. \tag{2.7}$$

Equation (2.7) implies that ℓ^y is $\mathbb{P}_x$-a.e. diffuse for any $x \in \mathbb{R}$ and has the invariance property (2.5). The following result is essentially from [7], see also [25].

Lemma 2.2. *Let $y \in \mathbb{R}$. Then $\mathbb{P}_y$ is the Palm measure of ℓ^y.*

Proof. Let $f\colon \Omega \times \mathbb{R} \to [0,\infty)$ be measurable. By definition (1.2) of $\mathbb{P}$ we have

$$\mathbb{E}\int f(\theta_s B, s)\, \ell^y(B, ds) = \int \mathbb{E}_0 \int f(\theta_s B + x, s)\, \ell^y(B + x, ds)\, dx.$$

By (2.6) this equals

$$\int \mathbb{E}_0 \int f(\theta_s B + x, s)\, \ell^{y-x}(B, ds)\, dx = \mathbb{E}_0 \iint f(\theta_s B - x + y, s)\, \ell^x(B, ds)\, dx,$$

where the equality comes from a change of variables and Fubini's theorem. By (2.4) this equals

$$\mathbb{E}_0 \int f(\theta_s B - B_s + y, s)\, ds.$$

Since $\mathbb{P}_0\{\theta_s B - B_s + y \in \cdot\} = \mathbb{P}_y$, we obtain

$$\mathbb{E}\int f(\theta_s B, s)\, \ell^y(B, ds) = \mathbb{E}_y \int f(B, s)\, ds \tag{2.8}$$

and hence the assertion. □

Equation (2.8) is the refined Campbell theorem (2.3) in the case $\xi = \ell^y$. In particular, it implies that ℓ^y has intensity 1:

$$\mathbb{E}\ell^y([0,1]) = 1. \tag{2.9}$$

3 Mass-stationarity

In this section we prove Theorem 1.1. Moreover, we show that the invariance property in Theorem 1.1 characterises mass-stationarity (defined below) of general diffuse random measures on the line.

Let ξ be a diffuse random measure on $\mathbb{R}$ and X be a random element in a space on which the additive group $\mathbb{R}$ acts measurably. Let $\theta_t(X, \xi)$ denote the shift of (X, ξ) by $t \in \mathbb{R}$. Let the pair (X, ξ) be defined as the identity on a canonical measurable space equipped with a σ-finite measure $\mathbb{Q}$. Let λ denote Lebesgue measure on $\mathbb{R}$. The pair (X, ξ) is called *mass-stationary* if, for all bounded Borel subsets C of $\mathbb{R}$ with $\lambda(C) > 0$ and $\lambda(\partial C) = 0$ and all nonnegative measurable functions f,

$$\mathbb{E}_{\mathbb{Q}} \iint \mathbf{1}_C(u) \frac{\mathbf{1}_{C-u}(s)}{\xi(C-u)} f(\theta_s(X, \xi), s+u)\, \xi(ds)\, du = \mathbb{E}_{\mathbb{Q}} \int \mathbf{1}_C(u) f((X, \xi), u)\, du. \tag{3.1}$$

Here we set $1/\xi(C-u) := 0$ whenever $\xi(C-u) = 0$. Mass-stationarity is a formalisation of the intuitive idea that the origin is a typical location in the mass of a random measure. The invariance property (3.1) can be interpreted probabilistically as saying that if the set C is placed uniformly at random around the origin and the origin shifted to a location chosen according to the mass distribution of ξ in that randomly placed set then the distribution of the pair (X, ξ) does not change.

The invariance property (3.2) in the following theorem is a new characterisation of mass-stationarity. It is similar to the well-known characterisation in the simple point process case (see e.g. [12, Theorem 11.4]) and is certainly more transparent than (3.1). It is however restricted to the diffuse case on the line while (3.1) works for general random measures in a group setting. The result (3.3) below is also new, but the equivalence of mass-stationarity and Palm measures was established in [14] for Abelian groups and in [13] for general locally compact groups.

Theorem 3.1. *Assume that* $\mathbb{Q}\{\xi(-\infty, 0) < \infty\} = \mathbb{Q}\{\xi(0, \infty) < \infty\} = 0$ *and let* S_r, $r \in \mathbb{R}$, *be the generalised inverse of the diffuse random measure* ξ *defined as in* (1.1). *Then*

$$\theta_{S_r}(X, \xi) \stackrel{d}{=} (X, \xi), \quad r \in \mathbb{R}, \tag{3.2}$$

if and only if (X, ξ) *is mass-stationary and if and only if the distribution* $\mathbb{Q}$ *of* (X, ξ) *is the Palm measure of* ξ *with respect to a* σ*-finite stationary measure* $\mathbb{Q}^*$. *The measure* $\mathbb{Q}^*$ *is uniquely determined by* $\mathbb{Q}$ *as follows: for each* $w > 0$ *and each bounded nonnegative measurable function* f,

$$\mathbb{E}_{\mathbb{Q}^*} f(X, \xi) = w^{-1} \mathbb{E}_{\mathbb{Q}} \int_0^{S_w} f(\theta_s(X, \xi))\, ds. \tag{3.3}$$

Proof. First assume (3.2). Then, $\mathbb{Q}\{\xi[0, \varepsilon] = 0\} = \mathbb{Q}\{\xi[S_1, S_1 + \varepsilon] = 0\} = 0$, for any $\varepsilon > 0$, where the second identity comes from $\xi[S_1, \infty) > 0$ $\mathbb{Q}$-a.e. and the definition of S_1. This easily implies that

$$S_r = -S_{-r} \circ \theta_{S_r} \quad \mathbb{Q}\text{-a.e., } r \in \mathbb{R}. \tag{3.4}$$

Let $C \subset \mathbb{R}$ be a bounded Borel with $\lambda(C) > 0$ and $\lambda(\partial C) = 0$. Changing variables and noting that, for any s in the support of ξ, we have $\xi(C - v + s) > 0$ for λ-a.e. $v \in C$, we obtain that the left-hand side of (3.1) equals

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}} \iint \mathbf{1}_C(v-s) \frac{\mathbf{1}_C(v)}{\xi(C-v+s)} f(\theta_s(X,\xi), v)\, \xi(ds)\, dv \\
&\quad = \mathbb{E}_{\mathbb{Q}} \iint \mathbf{1}_C(v - S_r) \frac{\mathbf{1}_C(v)}{\xi(C - v + S_r)} f(\theta_{S_r}(X,\xi), v)\, dr\, dv,
\end{aligned}$$

where we have changed variables to get the equality. The key observation (3.4) and assumption (3.2) yield that the above equals

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}} \iint \mathbf{1}_C(v + S_{-r}) \frac{\mathbf{1}_C(v)}{\xi(C-v)} f((X,\xi), v)\, dr\, dv \\
&\quad = \mathbb{E}_{\mathbb{Q}} \iint \mathbf{1}_C(v+s) \frac{\mathbf{1}_C(v)}{\xi(C-v)} f((X,\xi),v)\, \xi(ds)\, dv = \mathbb{E}_{\mathbb{Q}} \int \mathbf{1}_C(v) f((X,\xi),v)\, dv.
\end{aligned}$$

Thus (3.1) holds, that is, (X,ξ) is mass-stationary.

By [14, Theorem 6.3] equation (3.1) is equivalent to the existence of a stationary σ-finite measure $\mathbb{Q}^*$ such that $\mathbb{Q}$ is the Palm measure of ξ with respect to $\mathbb{Q}$. Mecke's [16] inversion formula (see also [14, Section 2]) implies that $\mathbb{Q}^*$ is uniquely determined by $\mathbb{Q}$ and that, moreover, $\mathbb{Q}^*\{\xi(-\infty,0) < \infty\} = \mathbb{Q}^*\{\xi(0,\infty) < \infty\} = 0$.

Fix $w > 0$. For the claim that $\mathbb{Q}^*$ defined by (3.3) is stationary when (3.2) holds, see Lemma 3.2 below. To show that $\mathbb{Q}$ is then the Palm measure of ξ with respect to this $\mathbb{Q}^*$ let f be nonnegative measurable and use (3.3) for the first step in the following calculation,

$$\begin{aligned}
w\, \mathbb{E}_{\mathbb{Q}^*} \int \mathbf{1}_{[0,1]}(s) f(\theta_s(X,\xi))\, \xi(ds) &= \mathbb{E}_{\mathbb{Q}} \iint \mathbf{1}_{[0,1]}(s) \mathbf{1}_{[0,S_w]}(t) f(\theta_s\theta_t(X,\xi))\, \theta_t\xi(ds)\, dt \\
&= \mathbb{E}_{\mathbb{Q}} \iint \mathbf{1}_{[0,1]}(v-t) \mathbf{1}_{[0,S_w]}(t) f(\theta_v(X,\xi))\, \xi(dv)\, dt \\
&= \mathbb{E}_{\mathbb{Q}} \iint \mathbf{1}_{[0,1]}(S_r - t) \mathbf{1}_{[0,S_w]}(t) f(\theta_{S_r}(X,\xi))\, dr\, dt \\
&= \mathbb{E}_{\mathbb{Q}} \iint \mathbf{1}_{[0,1]}(-S_{-r} - t) \mathbf{1}_{[0,S_w(\theta_{S_{-r}}(X,\xi))]}(t) f(X,\xi)\, dr\, dt \\
&= \mathbb{E}_{\mathbb{Q}} f(X,\xi) \iint \mathbf{1}_{[-1,0]}(u) \mathbf{1}_{[S_{-r},\, S_{-r+w}]}(u)\, dr\, du \\
&= w\, \mathbb{E}_{\mathbb{Q}} f(X,\xi),
\end{aligned}$$

where we have used (3.2) and (3.4) for the fourth identity and the final identity holds since the double integral equals w.

Finally, if $\mathbb{Q}$ is the Palm measure of ξ with respect to a σ-finite stationary measure $\mathbb{Q}^*$, then Theorem 2.1 implies (3.2) once we have shown for any $r \in \mathbb{R}$ that the allocation rule τ^r defined by $\tau^r(t) := S_r \circ \theta_t + t$ balances ξ with itself, that is,

$$\int \mathbf{1}\{\tau^r(s) \in \cdot\}\, \xi(ds) = \xi \quad \mathbb{Q}^*\text{-a.e.}$$

Asssume $r \geq 0$. Then, outside the $\mathbb{Q}^*$-null set $A := \{\xi(-\infty, 0) < \infty)\} \cup \{\xi(0, \infty) < \infty\}$ we obtain for any $a < b$ (interpreting $\xi[s, a]$ as $-\xi[a, s]$ for $s \geq a$) that

$$\begin{aligned}\int \mathbf{1}\{a \leq \tau^r(s) < b\}\, \xi(ds) &= \int \mathbf{1}\{s \leq b, \xi[s, a] \leq r, \xi[s, b] > r\}\, \xi(ds) \\ &= \int \mathbf{1}\{s \leq b, r < \xi[s, b] \leq r + \xi[a, b]\}\, \xi(ds) = \xi[a, b], \end{aligned}\tag{3.5}$$

which implies the desired balancing property. The case $r < 0$ can be treated similarly. $\square$

Lemma 3.2. *Let $S \geq 0$ be a random time and $\mathbb{Q}^*$ be the measure defined by setting, for each bounded nonnegative measurable function f,*

$$\mathbb{E}_{\mathbb{Q}^*} f(X, \xi) = \mathbb{E}_{\mathbb{Q}} \int_0^S f(\theta_s(X, \xi))\, ds.$$

If $\theta_S(X, \xi)$ has the same distribution as (X, ξ) under $\mathbb{Q}$ then (X, ξ) is stationary under $\mathbb{Q}^*$.

Proof. For each f as above and $t \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^*} f(\theta_t(X, \xi)) &= \mathbb{E}_{\mathbb{Q}} \int_t^{S+t} f(\theta_s(X, \xi))\, ds \\ &= \mathbb{E}_{\mathbb{Q}} \int_t^{S} f(\theta_s(X, \xi))\, ds + \mathbb{E}_{\mathbb{Q}} \int_S^{S+t} f(\theta_s(X, \xi))\, ds \\ &= \mathbb{E}_{\mathbb{Q}} \int_t^{S} f(\theta_s(X, \xi))\, ds + \mathbb{E}_{\mathbb{Q}} \int_0^{t} f(\theta_s(X, \xi))\, ds \\ &= \mathbb{E}_{\mathbb{Q}} \int_0^{S} f(\theta_s(X, \xi))\, ds = \mathbb{E}_{\mathbb{Q}^*} f(X, \xi),\end{aligned}$$

where the third identity follows from the assumption that $\theta_S(X, \xi)$ has the same distribution as (X, ξ) under $\mathbb{Q}$. $\square$

As a corollary we obtain an alternative construction of the stationary measure (2.1) by integrating over time rather than space.

Corollary 3.3. *Let $w > 0$ and consider T_w defined by (1.1). Then $\mathbb{P}^w = w\,\mathbb{P}$, where*

$$\mathbb{P}^w(A) := \mathbb{E}_0 \int_0^{T_w} \mathbf{1}\{\theta_s B \in A\}\, ds, \quad A \in \mathcal{A}.$$

Proof of Theorem 1.1. The result follows from Theorem 3.1 and Lemma 2.2. $\square$

The *invariant σ-algebra* is defined by

$$\mathcal{I} := \{A \in \mathcal{A} \colon \theta_t A = A \text{ for all } t \in \mathbb{R}\}.$$

We now apply Theorem 1.1 to prove the following result which we need in the proof of Theorem 1.3 in Section 5.

Theorem 3.4. *Let $A \in \mathcal{I}$. Then either $\mathbb{P}_x(A) = 0$ for any $x \in \mathbb{R}$ (in which case $\mathbb{P}(A) = 0$) or $\mathbb{P}_x(A^c) = 0$ for any $x \in \mathbb{R}$ (in which case $\mathbb{P}(A^c) = 0$).*

Proof. We first show that

$$\mathbb{P}_0(A) \in \{0, 1\}. \tag{3.6}$$

We use here the random times T_n (see (1.1)) for integers n. By Theorem 1.1, for any integer n, the processes $(B_{T_n - t})_{t\geq 0}$ and $(B_{T_n + t})_{t\geq 0}$ are independent one-sided Brownian motions. This implies that the processes

$$W_n := (B_{(T_n+t)\wedge(T_{n+1}-T_n)})_{t\geq 0},$$

are independent under $\mathbb{P}_0$. Since, by (2.5),

$$\inf\{t \geq 0 \colon \ell^0(\theta_{T_n} B, [0, t]) = 1\} = \inf\{t \geq 0 \colon \ell^0(B, [T_n, T_n + t]) = 1\} = T_{n+1} - T_n$$

holds $\mathbb{P}_0$-a.s. for any $n \in \mathbb{Z}$, the W_n have the distribution of a one-sided Brownian motion stopped at the time its local time at 0 reaches the value 1. Clearly we have that $B = F((W_n)_{n\in\mathbb{Z}})$ for a suitably defined measurable function F. By invariance of A and definition of the family $(W_n)_{n\in\mathbb{Z}}$,

$$\{F((W_n)_{n\in\mathbb{Z}}) \in A\} = \{B \in A\} = \{\theta_{T_1} B \in A\} = \{F((W_{n+1})_{n\in\mathbb{Z}}) \in A\},$$

where the final equation holds $\mathbb{P}_0$-a.s. Since iid-sequences are ergodic (by the law of large numbers), we obtain (3.6).

The refined Campbell theorem (2.8) implies (with λ denoting Lebesgue measure)

$$\mathbb{P}_x(A) = \lambda(C)^{-1}\mathbb{E}\mathbf{1}_A \ell^x(C), \quad x \in \mathbb{R}, \tag{3.7}$$

provided that $0 < \lambda(C) < \infty$. Assume now that $\mathbb{P}_0(A) = 0$. Then (3.7) implies that

$$\mathbb{P}(A \cap \{\ell^0(C) > 0\}) = 0$$

for all compact $C \subset \mathbb{R}$. Letting $C \uparrow \mathbb{R}$, we obtain $\mathbb{P}(A \cap \{\ell^0 \neq 0\}) = 0$, that is,

$$\mathbb{P}_x(A \cap \{\ell^0 \neq 0\}) = 0 \quad \lambda\text{-a.e. } x.$$

On the other hand, by (2.6), $\mathbb{P}_x\{\ell^0 \neq 0\} = \mathbb{P}_0\{\ell^{-x} \neq 0\} = 1$ for λ-a.e. x so that $\mathbb{P}_x(A) = 0$ for λ-a.e. x. Therefore $\mathbb{P}(A) = 0$. By (3.7) this implies $\mathbb{P}_x(A) = 0$ for all $x \in \mathbb{R}$. $\square$

4 Unbiased shifts and balancing allocation rules

In this section we prove a result that is more general than Theorem 1.2. Recall that $\mathbb{P}_\mu := \int \mathbb{P}_x \,\mu(dx)$ for a probability measure μ on $\mathbb{R}$.

Theorem 4.1. *Let T be a random time and μ, ν be probability measures on $\mathbb{R}$. Then the following two assertions are equivalent.*

(i) *It is true that*

$$\mathbb{P}_\mu\{\theta_T B - B_T \in \cdot\} = \mathbb{P}_0, \tag{4.1}$$

$\mathbb{P}_\mu\{B_T \in \cdot\} = \nu$, *and that* $\theta_T B - B_T$ *and* B_T *are independent under* $\mathbb{P}_\mu$.

(ii) *The allocation rule* τ_T *defined by* (1.5) *balances* ℓ^μ *and* ℓ^ν.

Proof. We start by noting that the random measures ℓ^μ and ℓ^ν are invariant in the sense of (2.2). This follows from the invariance of local time (2.5), (2.6) and Fubini's theorem.

Let us first assume that (i) holds. Then we have for any $A \in \mathcal{A}$ that

$$\begin{aligned}\mathbb{P}_\mu\{\theta_T B \in A\} &= \int \mathbb{P}_\mu\{\theta_T B - B_T + x \in A\}\,\nu(dx)\\ &= \int \mathbb{P}_0\{B + x \in A\}\,\nu(dx) = \mathbb{P}_\nu(A).\end{aligned}$$

Lemma 2.2 and Fubini's theorem imply that $\mathbb{P}_\nu$ is the Palm measure of ℓ^ν. Therefore we obtain from Theorem 2.1 that τ_T balances ℓ^μ and ℓ^ν.

Assume now that (ii) holds. By Theorem 2.1 we obtain for any $A \in \mathcal{A}$ that

$$\mathbb{P}_\mu\{\theta_T B \in A\} = \int \mathbb{P}_x(A)\nu(dx).$$

This implies

$$\mathbb{P}_\mu\{\theta_T B - B_T \in A', B_T \in C\} = \int_C \mathbb{P}_x\{B - x \in A'\}\,\nu(dx) = \mathbb{P}_0(A')\nu(C)$$

for any $A' \in \mathcal{A}$ and any $C \in \mathcal{B}$. This yields (i). □

Remark 4.2. An *extended* allocation rule is a mapping $\tau\colon \Omega \times \mathbb{R} \to [0,\infty]$ that has the equivariance property (1.3). The balancing property (1.4) can then be defined as before. Using these concepts, Theorem 4.1 can be proved for a subprobability measure $\nu \neq 0$. The conditions in (i) have to be replaced with $\mathbb{P}_\mu\{\theta_T B - B_T \in \cdot \mid T < \infty\} = \mathbb{P}_0$, $\mathbb{P}_\mu\{T < \infty, B_T \in \cdot\} = \nu$ and the independence of $\theta_T B - B_T$ and B_T under $\mathbb{P}_\mu\{\cdot | T < \infty\}$.

5 Existence of unbiased shifts

In this section we prove Theorem 1.3. The proof is based on the following new balancing result for general random measures on the line, which is inspired by [8].

Theorem 5.1. *Let* ξ *and* η *be jointly stationary orthogonal diffuse random measures on* $\mathbb{R}$ *with finite intensities. Assume further that*

$$\mathbb{E}\big[\xi[0,1]\big|\mathcal{I}\big] = \mathbb{E}\big[\eta[0,1]\big|\mathcal{I}\big] \quad \mathbb{P}\text{-}a.e.$$

Then the mapping $\tau\colon \Omega \times \mathbb{R} \to \mathbb{R}$, *defined by*

$$\tau(s) := \inf\{t > s\colon \xi[s,t] = \eta[s,t]\}, \quad s \in \mathbb{R}, \tag{5.1}$$

is an allocation rule balancing ξ *and* η.

Remark 5.2. For convenience, Theorem 5.1 has been formulated for random measures defined on our canonical $(\Omega, \mathcal{A}, \mathbb{P})$. However, the special structure of that space is of no importance here.

We start the proof of Theorem 5.1 with an analytic lemma. Here and later it is convenient to work with the continuous function $f\colon \mathbb{R} \to \mathbb{R}$, defined by

$$f(t) := \begin{cases} \xi[0,t] - \eta[0,t], & \text{if } t \geq 0, \\ \eta[t,0] - \xi[t,0], & \text{if } t < 0. \end{cases}$$

Lemma 5.3. *Suppose ξ and η are orthogonal diffuse measures. Then*

$$\int \mathbf{1}\{\tau(s) \in \cdot\}\, \xi(ds) = \eta(\cdot) \quad \textit{on } [0,a],$$

provided that $f(t) \geq 0$ for all $t \in (0,a)$.

The proof of Lemma 5.3 rests on three further lemmas.

Lemma 5.4.

(a) For ξ-almost every s there exists $s_n \downarrow s$ with $f(s_n) > f(s)$.

(b) For η-almost every s there exists $s_n \downarrow s$ with $f(s_n) < f(s)$.

Proof. It suffices to prove (a), as (b) follows by reversing the roles of ξ and η. Recall that ξ and η are orthogonal and hence there exists a Borel set A with $\eta(A) = 0$ and $\xi(A^{\mathrm{c}}) = 0$. We need to show that, for each $\epsilon > 0$,

$$\xi(A_\epsilon) = 0 \text{ where } A_\epsilon := \{s \in A\colon f(t) \leq f(s) \text{ for all } t \in [s, s+\epsilon)\}.$$

Given any $\delta > 0$ we may choose an open set $O \supset A$ with $\eta(O) < \delta$. We can cover A_ϵ by a countable collection $\mathcal{I}$ of nonoverlapping intervals $[s, s+\epsilon_s]$, $s \in A_\epsilon$, $0 < \epsilon_s \leq \epsilon$, such that $(s, s+\epsilon_s) \subset O$. Indeed, suppose that O' is a connected component of O, which intersects A_ϵ. If there is a minimal element s in $O' \cap A_\epsilon$ let ϵ_s be the minimum of ϵ and the distance of s to the right endpoint of O'. Add the interval $[s, s+\epsilon_s]$ to the collection $\mathcal{I}$ and remove it from A_ϵ and O. If no such minimum exists we can pick a strictly decreasing sequence $s_n \in O' \cap A_\epsilon$, $n \in \mathbb{N}$, converging to the infimum. Let ϵ_{s_1} be the minimum of ϵ and the distance of s_1 to the right endpoint of O', and, for $i \geq 2$, let ϵ_{s_i} be the minimum of ϵ and $s_{i-1} - s_i$. Add all intervals $[s_i, s_i + \epsilon_{s_i}]$ to the collection $\mathcal{I}$ and remove their union from A_ϵ and O. Note that after one such step (performed in every connected component) all of A_ϵ in connected components of length at most ϵ will be removed, and the lower bound of the intersection of all other connected components with A_ϵ, if finite, is increased by at least ϵ. Also, after one step, the intersection of any connected component with A_ϵ is either empty or bounded from below. Therefore, every set of the form $[-M, M] \cap A_\epsilon$ will be completely covered after finitely many steps by nonoverlapping intervals, as required. Observe that $\xi(I) \leq \eta(I)$ for every interval in the collection, and hence

$$\xi(A_\epsilon) \leq \sum_{I \in \mathcal{I}} \xi(I) \leq \sum_{I \in \mathcal{I}} \eta(I) \leq \eta(O) \leq \delta.$$

The result follows as $\delta > 0$ was arbitrary. $\square$

We now fix $a \geq 0$ and decompose f on $[0, a]$ according to its backwards running minimum m given by

$$m(t) = \min\{f(s) \colon t \leq s \leq a\},$$

see Figure 1 for illustration. The nonegative function $f - m$ can be decomposed on $[0, a]$ into a family $\mathcal{E}$ of excursions $e \colon [0, \infty) \to [0, \infty)$ with starting times $t_e \in [0, a]$. Note that an excursion $e \colon [0, \infty) \to [0, \infty)$ is a function such that there exists a number $\sigma_e > 0$, called the lifetime of the excursion, such that $e(0) = 0$, $e(s) > 0$ for all $0 < s < \sigma_e$, and $e(s) = 0$ for all $s \geq \sigma_e$. Formally putting $e(s) = 0$ for all $s < 0$ the decomposition can be written as

$$f(t) - m(t) = \sum_{(e,t_e) \in \mathcal{E}} e(t - t_e).$$

Note that the intervals $(t_e, t_e + \sigma_e)$, $(e, t_e) \in \mathcal{E}$, are disjoint. We denote by C the complement of their union in $[0, a]$, i.e. $C = \{t \in [0, a] \colon f(t) = m(t)\}$.

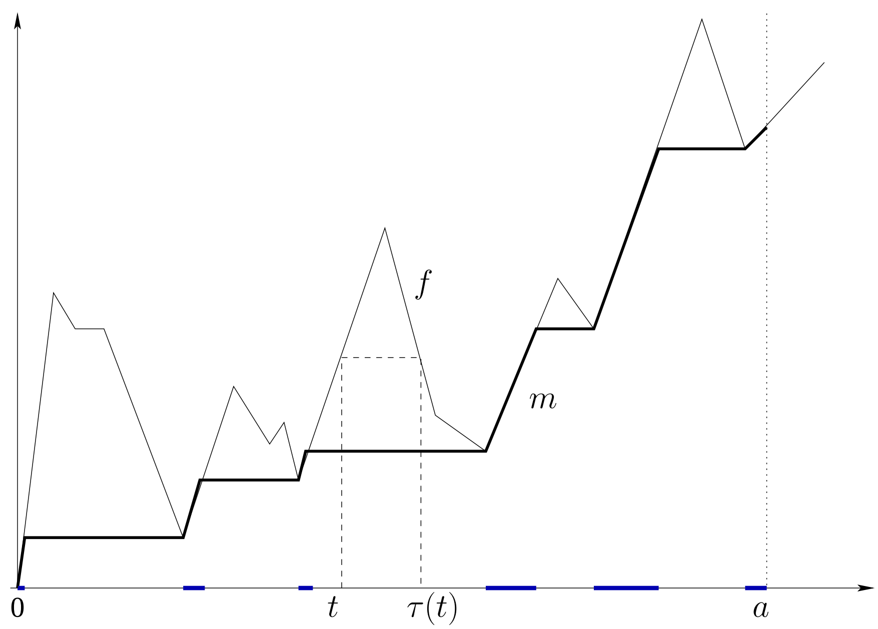

Figure 1: A schematic picture of the function f and its backwards running minimum m, in bold. The set C is marked bold on the abscissa, and an example of the mapping $t \mapsto \tau(t)$ indicated by dashed lines.

Lemma 5.5. *For every $(e, t_e) \in \mathcal{E}$ we have*

$$\xi\{s \in (t_e, t_e + \sigma_e) \colon \tau(s) \leq a\} = \xi(t_e, t_e + \sigma_e) = \eta(t_e, t_e + \sigma_e).$$

Proof. We only have to show that $\tau(s) \leq a$ for ξ-almost every $s \in (t_e, t_e + \sigma_e)$. By Lemma 5.4 (a), for ξ-almost every $s \in (t_e, t_e + \sigma_e)$, there exists $s_n \downarrow s$ such that $f(s_n) > f(s)$. As $f(s) > f(t_e + \sigma_e)$, by continuity of f, we infer that there exists $s^* \in (s, t_e + \sigma_e)$ such that $f(s^*) = f(s)$. Therefore $\tau(s) \leq s^* \leq a$ as required. $\square$

Lemma 5.6. *We have*

$$\xi\{s \in C \colon \tau(s) \le a\} = \eta(C) = 0.$$

Proof. First observe that if $s \in C$, then $f(t) \ge f(s)$ for all $t \in (s, a]$. If $f(t) > f(s)$ for all $t \in (s, a]$ then $\tau(s) > a$. Otherwise there exists a maximal $t \in (s, a]$ with $f(s) = f(t)$. Then $f(t)$ is a true local minimum of f in the sense that there exists $r > 0$ with $f(s) \ge f(t)$ for all $s \in (t - r, t)$ and $f(s) > f(t)$ for all $s \in (t, (t + r) \wedge a)$. In particular there are at most countably many levels $f(s)$ where this can happen. Fixing such a level l we note that $\xi\{s \in C \colon f(s) = l\} = \eta\{s \in C \colon f(s) = l\}$. Summing over all these levels we see that $\xi\{s \in C \colon \tau(s) \le a\} \le \eta(C)$. We conclude the proof by showing that $\eta(C) = 0$. Lemma 5.4 (b) ensures that, for η-almost every $s \in [0, a]$ there exists $s_n \downarrow s$ such that $f(s_n) < f(s)$, which implies that $s \notin C$. Hence the stated equality follows. □

Proof of Lemma 5.3. Taking the sum over the equations in the previous two lemmas we obtain $\xi\{s \ge 0 \colon \tau(s) \le a\} = \eta[0, a]$. This implies

$$\int \mathbf{1}\{0 \le \tau(s) \le a\}\, \xi(ds) = \eta[0, a], \quad a \ge 0,$$

as any $s < 0$ with $f(s) < 0$ satisfies $\tau(s) \notin [0, a]$, and Lemma 5.4 (a) implies that ξ-almost every $s < 0$ with $f(s) \ge 0$ satisfies $\tau(s) < 0$, and so $\xi\{s < 0 \colon 0 < \tau(s) \le a\} = 0$. □

Proof of Theorem 5.1. Define

$$\begin{aligned}
\xi^\infty &:= \int \mathbf{1}\{\tau(s) = \infty, s \in \cdot\}\, \xi(ds), \\
\eta^* &:= \int \mathbf{1}\{\tau(s) \in \cdot\}\, \xi(ds).
\end{aligned}$$

Recall that by Lemma 5.3 we have $\eta^* = \eta$ on $[s, \infty)$ provided $\xi[s, t] \ge \eta[s, t]$ for all $t \ge s$. By Lemma 5.4 this holds for ξ^∞-a.e. s. Moreover, by stationarity of ξ^∞ we have that $\mathbb{P}\{\xi^\infty(-\infty, s] = 0, \xi^\infty \ne 0\} = 0$ for all $s \in \mathbb{R}$. We infer that

$$\eta^* = \eta \quad \mathbb{P}\text{-a.e. on } \{\xi^\infty \ne 0\}. \tag{5.2}$$

Using the refined Campbell theorem (2.3) twice, we obtain

$$\begin{aligned}
\mathbb{E}\mathbf{1}\{\xi^\infty \ne 0\} &\int_0^1 \mathbf{1}\{\tau(s) < \infty\}\, \xi(ds) \\
&= \mathbb{E} \int_0^1 \mathbf{1}\{\xi^\infty \circ \theta_s \ne 0, \tau(\theta_s, 0) < \infty\}\, \xi(ds) \\
&= \mathbb{Q}_\xi\big\{\xi^\infty \ne 0, \tau(0) < \infty\big\} \\
&= \mathbb{E}_{\mathbb{Q}_\xi} \int \mathbf{1}\{\xi^\infty \ne 0, \tau(0) + s \in [0, 1]\}\, ds \\
&= \mathbb{E} \int \mathbf{1}\{\xi^\infty \ne 0, \tau(s) \in [0, 1]\}\, \xi(ds) \\
&= \mathbb{E}\mathbf{1}\{\xi^\infty \ne 0\}\, \eta^*[0, 1].
\end{aligned} \tag{5.3}$$

Using first (5.2) and then our assumption gives

$$\mathbb{E}\mathbf{1}\{\xi^\infty \neq 0\}\eta^*[0,1] = \mathbb{E}\mathbf{1}\{\xi^\infty \neq 0\}\eta[0,1] = \mathbb{E}\mathbf{1}\{\xi^\infty \neq 0\}\xi[0,1],$$

and together with (5.3) we infer that

$$\mathbb{E}\mathbf{1}\{\xi^\infty \neq 0\} \int_0^1 \mathbf{1}\{\tau(s) < \infty\} \, \xi(ds) = \mathbb{E}\mathbf{1}\{\xi^\infty \neq 0\} \, \xi[0,1],$$

and therefore $\tau(s) < \infty$ for ξ-a.e. s, $\mathbb{P}$-a.e. In particular, this implies that τ is a well-defined allocation rule. An analogous argument implies that

$$\tau^{-1}(s) > -\infty \quad \eta\text{-a.e. } s, \ \mathbb{P}\text{-a.e.},$$

where

$$\tau^{-1}(s) = \sup\{t < s \colon \xi[t,s] = \eta[t,s]\}$$

is the inverse of τ. We now use this to show that τ balances ξ and η. Fixing $a < b$ we aim to show that $\eta^*[a,b] = \eta[a,b]$. If $f(t) \geq f(a)$ for all $t \in [a,b]$ this holds by Lemma 5.3. Otherwise we apply this lemma to suitably chosen alternative intervals. To this end let

$$a^* := \min\{s \in [a,b] \colon f(s) \leq f(t) \text{ for all } a \leq t \leq b\}$$

be the leftmost minimiser of f on $[a,b]$. As $\eta(a^* - \frac{1}{n}, a^*] \geq f(a^* - \frac{1}{n}) - f(a^*) > 0$ for all sufficiently large $n \in \mathbb{N}$, we find a decreasing sequence s_n with $\tau(s_n) \downarrow a^*$ and hence $f(s_n) \to f(a^*)$. Then $s_n \downarrow s \in [-\infty, a]$ and $f(s) = f(a^*)$ if $s \neq -\infty$.

Assuming first that $s \neq -\infty$ we obtain from Lemma 5.3 that

$$\int \mathbf{1}\{\tau(s) \in \cdot\} \, \xi(ds) = \eta(\cdot) \quad \text{on } [s,b],$$

which implies the statement. Now assume that $s = -\infty$. In this case we get $\eta^* = \eta$ on $[s_n, \tau(s_n)]$ and on $[a^*, b]$ for every n, and the result follows as $n \to \infty$. $\square$

Proof of Theorem 1.3. By Theorem 3.4, $\mathbb{E}[\ell^0[0,1]|\mathcal{I}] = \mathbb{E}[\ell^\nu[0,1]|\mathcal{I}]$ almost surely. Since ℓ^0 and ℓ^ν are orthogonal, we can combine Theorems 5.1 and 1.2 to obtain the result. $\square$

Remark 5.7. Assume in Theorem 1.3 that ν is a subprobability measure. Then T takes the value ∞ with positive $\mathbb{P}_0$-probability. Indeed, by Remark 4.2, defining the extended allocation rule τ by $\tau(s) := s + T \circ \theta_s$ we get that τ balances the restriction of ℓ^0 to $\{s \colon \tau(s) < \infty\}$ and ℓ^ν. Assertion (i) of Theorem 4.1 remains valid in the sense explained in Remark 4.2 and gives $\mathbb{P}_0\{T < \infty\} = \nu(\mathbb{R})$.

Now assume in Theorem 1.3 that ν is a locally finite measure with $\nu(\mathbb{R}) > 1$. Then $\mathbb{P}_0\{T < \infty\} = 1$ and τ balances ℓ^0 and $\eta := \int \mathbf{1}\{\tau(s) \in \cdot\} \, \ell^0(ds)$. The proof of Theorem 5.1 still yields the inequality $\eta \leq \ell^\nu$. In particular η is a diffuse (and invariant) random measure and must therefore be of the form $\eta = \ell^{\nu'}$ for some measure ν', see e.g. [12, Theorem 22.25]. In fact, ν' is a probability measure smaller than ν. Hence T is an unbiased shift embedding ν'. Some properties of ν' can be found in [3].

The nonnegative unbiased shifts in Theorem 1.3, Theorem 1.4 and in Theorem 1.1 are all stopping times. In the next example we construct a nonnegative unbiased shift embedding a distribution not concentrated at zero, which is not a stopping time.

Example 5.8. Let $x \in \mathbb{R} \setminus \{0\}$. We define an allocation rule τ that balances ℓ^0 and ℓ^x and such that $T := \tau(0)$ is nonnegative but not a stopping time. The mapping τ is the composition of the following five allocation rules. Let $\tau_1 = \tau_4$ balance ℓ^0 and ℓ^x according to Theorem 1.3. Let τ_2 balance ℓ^x and ℓ^x by shifting forward one mass-unit, that is , let $\tau_2(0)$ be defined by (1.1) with $r = 1$ and with ℓ^0 replaced with ℓ^x. Let τ_3 balance ℓ^x and ℓ^0 according to Theorem 1.3. Finally define τ_5 by shifting *backward* one mass-unit in the local time at x, that is, let τ_5 be defined by (1.1) with $r = -1$ and ℓ^0 replaced with ℓ^x. The composition τ of these allocation rules balances ℓ^0 and ℓ^x. Moreover, $T := \tau(0) \geq \tau_1(0) \geq 0$. However, T is not a stopping time. This example can be extended to a general target distribution ν.

6 Target distributions with an atom at zero

In this section we prove Theorems 1.4, 1.5, and 1.6. In contrast to the previous section we allow here for an atom at 0.

Proof of Theorem 1.4. Let $y \in \mathbb{R} \setminus \{0\}$ such that $\nu\{y\} = 0$ and define

$$\mu := \nu - \nu\{0\}\delta_0 + \nu\{0\}\delta_y.$$

Theorems 1.2 and 1.3 imply that the allocation rule

$$\tau'(s) := \inf\big\{t > s\colon \ell^0[s,t] = \ell^\mu[s,t]\big\}, \quad s \in \mathbb{R},$$

balances ℓ^0 and ℓ^μ. The same theorems imply that there is an allocation rule τ'' that balances ℓ^y and ℓ^0. Define

$$\tau(s) := \begin{cases} \tau'(s), & \text{if } B_{\tau'(s)} \neq y, \\ \tau''(\tau'(s)), & \text{if } B_{\tau'(s)} = y. \end{cases}$$

Then we have for any Borel set $C \subset \mathbb{R}$ outside a fixed $\mathbb{P}$-null set that

$$\begin{aligned}
&\int \mathbf{1}\{\tau(s) \in C\}\, \ell^0(ds) \\
&\quad = \int \mathbf{1}\{\tau(s) \in C, B_{\tau'(s)} \neq y\}\, \ell^0(ds) + \int \mathbf{1}\{\tau(s) \in C, B_{\tau'(s)} = y\}\, \ell^0(ds) \\
&\quad = \int \mathbf{1}\{\tau'(s) \in C, B_{\tau'(s)} \neq y\}\, \ell^0(ds) + \int \mathbf{1}\{\tau''(\tau'(s)) \in C, B_{\tau'(s)} = y\}\, \ell^0(ds) \\
&\quad = \int \mathbf{1}\{s \in C, B_s \neq y\}\, \ell^\mu(ds) + \int \mathbf{1}\{\tau''(s) \in C, B_s = y\}\, \ell^\mu(ds) \\
&\quad = \int \mathbf{1}\{x \neq 0\}\mathbf{1}\{s \in C\}\, \ell^x(ds)\, \nu(dx) + \nu\{0\} \int \mathbf{1}\{\tau''(s) \in C\}\, \ell^y(ds) \\
&\quad = \int \mathbf{1}\{x \neq 0\}\mathbf{1}\{s \in C\}\, \ell^x(ds)\, \nu(dx) + \nu\{0\}\ell^0(C) = \ell^\nu(C),
\end{aligned}$$

where we have used (2.7) (and $\nu\{y\} = 0$) in the penultimate equation. Hence τ balances ℓ^0 and ℓ^ν. Theorem 1.2 now implies that $T := \tau(0)$ is an unbiased shift embedding ν. $\square$

Proof of Theorem 1.5. let T be any unbiased shift embedding ν and define $\tau := \tau_T$. Outside a fixed $\mathbb{P}$-null set we obtain for any Borel set $C \subset \mathbb{R}$ that

$$\begin{aligned}
&\int \mathbf{1}\{s \in C, \tau(s) = s\}\, \ell^0(ds) = \int \mathbf{1}\{\tau(s) \in C, \tau(s) = s\}\, \ell^0(ds) \\
&\quad = \int \mathbf{1}\{\tau(s) \in C, \tau(s) = s, B_{\tau(s)} = 0\}\, \ell^0(ds) \\
&\quad \le \int \mathbf{1}\{\tau(s) \in C, B_{\tau(s)} = 0\}\, \ell^0(ds) = \int \mathbf{1}\{s \in C, B_s = 0\}\, \ell^\nu(ds) = \nu\{0\}\ell^0(C),
\end{aligned}$$

where we have used (2.7) to obtain the final identity. This implies that

$$\mathbf{1}\{\tau(s) = s\} \le \nu\{0\} \quad \ell^0\text{-a.e. } s,\ \mathbb{P}\text{-a.e.}$$

Assuming now that $\nu\{0\} < 1$ we obtain $\tau(s) \ne s$ for ℓ^0-a.e. s, $\mathbb{P}$-almost everywhere. Lemma 2.2 now implies (1.7). $\square$

Proof of Theorem 1.6. Let $\tau' := \tau_{T'}$, where T' is given by (1.6) with $\nu = p\delta_1 + (1-p)\delta_2$. Define a stationary random measure ξ by $\xi(dt) := \mathbf{1}\{B_{\tau'(t)} = 2\}\, \ell^0(dt)$. The allocation rule

$$\tau''(s) := \inf\big\{t > s \colon \xi[s,t] = 1\big\}$$

balances ξ with itself. Define

$$\tau(s) := \begin{cases} s, & \text{if } B_{\tau'(s)} = 1, \\ \tau''(s), & \text{if } B_{\tau'(s)} = 2. \end{cases}$$

It is easy to see that τ balances ℓ^0 with itself. Lemma 2.2 and Theorem 1.2 (or a direct calculation) implies that $T := \tau(0)$ satisfies

$$\mathbb{P}_0\{T = 0\} = \mathbb{P}\{B_{\tau'(0)} = 0\} = p.$$

Since T is an unbiased shift, the proof is complete. $\square$

7 Stability and minimality of balancing allocations

The following definition is a one-sided version of the notion of stability introduced in [8] for point processes. We call an allocation rule $\tau\colon \Omega \times \mathbb{R} \to \mathbb{R}$ balancing ξ and η *right-stable* if $\tau(s) \ge s$ for all $s \in \mathbb{R}$ and

$$\xi \otimes \xi\big\{(s,t)\colon t < s \le \tau(t) < \tau(s)\big\} = 0 \quad \mathbb{P}\text{-a.e.}$$

Roughly speaking this means that the mass of pairs (s,t) such that s would prefer the partner of t over its own partner, while $\tau(t)$ would prefer s over t as a partner, vanishes.

Theorem 7.1. *Let ξ and η be invariant random measures satisfying the conditions of Theorem 5.1, and suppose $\tau\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is the allocation rule constructed in the theorem. Then τ is right-stable.*

Proof. By Lemma 5.4 (a) and continuity of f, we have for ξ-a-e. s that $f(s) < f(r)$ for all $r \in (s, \tau(s))$. Hence $\xi \otimes \xi$-almost every pair (s,t) with $t < s \le \tau(t) < \tau(s)$ satisfies $f(t) < f(s) < f(\tau(t))$ contradicting the definition of τ. $\square$

Right-stable allocation rules have a useful minimality property.

Theorem 7.2. *Any right-stable allocation rule τ balancing two measures ξ and η is minimal in the sense that if σ is another allocation rule balancing ξ and η such that $s \le \sigma(s) \le \tau(s)$ for ξ-almost every $s \in \mathbb{R}$, then $\xi\{s\colon \sigma(s) < \tau(s)\} = 0$.*

Proof. By right-stability of τ we have, *for ξ-almost every a,*

$$s \in [a,\tau(a)] \Longleftrightarrow \tau(s) \in [a,\tau(a)] \quad \xi\text{-a.e. } s. \tag{7.1}$$

From the assumption $s \le \sigma(s) \le \tau(s)$ and (7.1) we obtain for any $t \in [a,\tau(a)]$ that $\tau(s) \in [a,t]$ implies $\sigma(s) \in [a,t]$ for ξ-almost every s. Therefore

$$\eta[a,t] = \int \mathbf{1}\{\tau(s) \in [a,t]\}\,\xi(ds) \le \int \mathbf{1}\{\sigma(s) \in [a,t]\}\,\xi(ds) = \eta[a,t].$$

This implies

$$\mathbf{1}\{\tau(s) \in [a,t]\} = \mathbf{1}\{\sigma(s) \in [a,t]\} \quad \xi\text{-a.e. } s \in \mathbb{R}.$$

Therefore τ and σ coincide ξ-almost everywhere on $\tau^{-1}([a,\tau(a)])$.

Now fix some $b \in \mathbb{R}$ and recall the definition of the backwards running minimum $m(t) = \min\{f(s)\colon t \le s \le b\}$ and the set $C = \{t \le b\colon m(t) = f(t)\}$. We have seen that the complement of C consists of countably many intervals $(a,\tau(a))$ as above and therefore τ and σ coincide ξ-almost everywhere on $\tau^{-1}((-\infty,b]\setminus C)$. On the other hand, by Lemma 5.6 we have $\xi(\tau^{-1}(C)) = \eta(C) = 0$, as required to finish the argument. $\square$

Remark 7.3. One could define an allocation rule τ to be *stable* if

$$\xi \otimes \xi\big\{(s,t)\colon |s-\tau(t)| < |s-\tau(s)|, |s-\tau(t)| < |t-\tau(t)|\big\} = 0.$$

The rule τ of Theorem 5.1 does not satisfy this. We do not know if stable allocation rules in the above sense exist, or if they are unique.

An unbiased shift T is called *minimal unbiased shift* if $\mathbb{P}_0\{T \ge 0\} = 1$ and if any other unbiased shift S such that $\mathbb{P}_0\{0 \le S \le T\} = 1$ and $\mathbb{P}_0\{B_T \in \cdot\} = \mathbb{P}_0\{B_S \in \cdot\}$ satisfies $\mathbb{P}_0\{S = T\} = 1$. The following theorem provides more insight into the set of all minimal unbiased shifts. The result and its proof are motivated by Proposition 2 in [18].

Theorem 7.4. *Let T be an unbiased shift embedding the probability measure ν and such that $\mathbb{P}_0\{T \ge 0\} = 1$. Then there exists a minimal unbiased shift T^* embedding ν and such that $\mathbb{P}_0\{0 \le T^* \le T\} = 1$.*

Proof. Let $\mathcal{T}$ denote the set of all unbiased shifts S embedding ν and such that $\mathbb{P}_0\{0 \le S \le T\} = 1$. This is a partially ordered set, where we do not distinguish between elements that coincide $\mathbb{P}_0$-a.s. By the Hausdorff maximal principle (see, e.g. [5, Section 1.5]) there is a *maximal chain* $\mathcal{T}' \subset \mathcal{T}$. This is a totally ordered set that is not contained in a strictly bigger totally ordered set. Let

$$\alpha := \sup_{S\in\mathcal{T}'} \mathbb{E}_0 e^{-S}.$$

Then there is a sequence S_n, $n \in \mathbb{N}$, such that $\mathbb{E}_0 e^{-S_n} \to \alpha$ as $n \to \infty$. Since $\mathcal{T}'$ is totally ordered it is no restriction of generality to assume that the S_n are decreasing $\mathbb{P}_0$-a.s. Define $T^* := \lim_{n\to\infty} S_n$. By construction and monotone convergence

$$\mathbb{E}_0 e^{-T^*} = \alpha. \tag{7.2}$$

We also note that $\mathbb{P}_0\{0 \le T^* \le T\} = 1$.

We claim that T^* is a minimal unbiased shift embedding ν and first show that T^* is an unbiased shift. Let $k \in \mathbb{N}$, and consider continuous and bounded functions $f\colon \mathbb{R}^k \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$. Let $t_1, \dots, t_k \in \mathbb{R}$. Since $S_n \in \mathcal{T}$ for any $n \in \mathbb{N}$ we have that

$$\mathbb{E}_0 f(B_{S_n+t_1} - B_{S_n}, \dots, B_{S_n+t_k} - B_{S_n}) g(B_{S_n}) = \mathbb{E}_0 f(B_{t_1}, \dots, B_{t_k}) \int g(x)\, \nu(dx). \tag{7.3}$$

By bounded convergence the above left-hand side converges towards

$$\mathbb{E}_0 f(B_{T^*+t_1} - B_{T^*}, \dots, B_{T^*+t_k} - B_{T^*}) g(B_{T^*})$$

as $n \to \infty$. The monotone class theorem implies that T^* is an unbiased shift embedding ν.

It remains to show the minimality property of T^*. Assume on the contrary that there is some unbiased shift S embedding ν such that $\mathbb{P}_0\{0 \le S \le T^*\} = 1$ and $\mathbb{P}_0\{S < T^*\} > 0$. The last two relations imply that

$$\mathbb{E}_0 e^{-S} > \mathbb{E}_0 e^{-T^*}.$$

By (7.2) this means that $S \notin \mathcal{T}'$. On the other hand, since $\mathbb{P}_0\{S \le T^* \le T\} = 1$, we have that $S \in \mathcal{T}$, contradicting the maximality property of $\mathcal{T}'$. $\square$

As announced in the introduction the stopping time (1.6) is a minimal unbiased shift:

Theorem 7.5. *Let ν be a probability measure on $\mathbb{R}$ with $\nu\{0\} = 0$. Then T defined by* (1.6) *is a minimal unbiased shift.*

Proof. Let S be an unbiased shift embedding ν and such that $\mathbb{P}_0\{0 \le S \le T\} = 1$. Theorem 1.2 implies that the allocation rules τ_S and τ_T balance ℓ^0 and ℓ^ν. By Theorem 7.1, τ_T is right-stable $\mathbb{P}$-a.e. The assumptions yield $\ell^0\{s\colon s \le \tau_S(s) \le \tau_T(s)\} = 0$ $\mathbb{P}$-a.e. By Theorem 7.2 we therefore have $\ell^0\{s\colon \tau_S(s) < \tau_T(s)\} = 0$ $\mathbb{P}$-a.e. This readily implies that $\mathbb{P}_0\{S = T\} = 1$. $\square$

8 Moments of unbiased shifts

In this section we discuss moment properties of unbiased shifts. The following two theorems together were stated as Theorem 1.7 in the introduction.

Theorem 8.1. *Suppose ν is a target distribution with $\nu\{0\} = 0$, and $\int |x| \, \nu(dx) < \infty$, and the stopping time $T \geq 0$ is an unbiased shift embedding ν. Then*

$$\mathbb{E}_0 T^{1/4} = \infty.$$

Proof. We start the proof with a reminder of the Barlow-Yor inequality [2], which states that, for any $p > 0$ there exist constants $0 < c < C$ such that, for all stopping times T,

$$c \, \mathbb{E}_0 T^{p/2} \leq \mathbb{E} \sup_x \ell^x[0, T]^p \leq C \, \mathbb{E}_0 T^{p/2}.$$

Hence it suffices to verify that $\mathbb{E}\ell^0[0, T]^{1/2} = \infty$.

The proof of this fact uses an argument similar to that in the proof of Theorem 2 in [8]. Let $\tau = \tau_T$ be the allocation rule associated with T and set $T_t = \inf\{s > 0 \colon \ell^0[0, s] = t\}$. Then, on the one hand,

$$\begin{aligned}
\mathbb{E}_0 \int \mathbf{1}\{0 \leq s \leq T_t, \tau(s) \notin [0, T_t]\} \, \ell^0(ds) &= \mathbb{E}_0 \int_0^{T_t} \mathbf{1}\{\tau(s) - s > T_t - s\} \, \ell^0(ds) \\
&= \int_0^t \mathbb{P}_0\{\tau(T_s) - T_s > T_t - T_s\} \, ds = \int_0^t \mathbb{P}_0\{T \circ \theta_{T_s} > T_{t-s} \circ \theta_{T_s}\} \, ds \\
&= \int_0^t \mathbb{P}_0\{T > T_s\} \, ds = \mathbb{E}_0[\ell^0[0, T] \wedge t],
\end{aligned}$$

where we have used the strong Markov property at T_s (or Theorem 1.1) for the fourth step and change of variable for the second and fifth steps. On the other hand, the fact that τ balances ℓ^0 and ℓ^ν easily implies that

$$\int \mathbf{1}\{0 \leq s \leq T_t, \tau(s) \notin [0, T_t]\} \, \ell^0(ds) \geq (\ell^0[0, T_t] - \ell^\nu[0, T_t])_+.$$

Hence, combining these two facts,

$$\mathbb{E}_0[\ell^0[0, T] \wedge t] \geq \mathbb{E}_0(\ell^0[0, T_t] - \ell^\nu[0, T_t])_+.$$

Observe that $\ell^0[0, T_t] = t = \mathbb{E}\ell^\nu[0, T_t]$, where the second identity is a trivial consequence of the second Ray-Knight theorem (see Theorem 2.3 in Chapter XI of [21]) and in fact a consequence of general Palm theory. Jensen's inequality and again the second Ray-Knight theorem imply that

$$\mathbb{E}_0(\ell^\nu[0, T_1])^2 \leq \mathbb{E}_0 \int (\ell^x[0, T_1])^2 \, \nu(dx) = \int (1 + |x|) \, \nu(dx) \tag{8.1}$$

which is finite by assumption. Hence, by the central limit theorem,

$$\liminf_{t \to \infty} t^{-1/2} \mathbb{E}_0(\ell^0[0, T_t] - \ell^\nu[0, T_t])_+ =: c_1 > 0.$$

Assume for contradiction that $\mathbb{E}[\ell^0[0, T]^{1/2}] < \infty$. Since $t^{-1/2}(\ell^0[0, T] \wedge t) \leq \ell^0[0, T]^{1/2}$, dominated convergence implies that $t^{-1/2}\mathbb{E}_0[\ell^0[0, T] \wedge t] \to 0$ as $t \to \infty$, contradicting the positivity of c_1. ∎

Note that the unbiased shifts defined in (1.6) satisfy the conditions of Theorem 8.1 if ν has finite mean. The next result shows that they have nearly optimal moment properties.

Theorem 8.2. *Let ν satisfy $\int |x|\, \nu(dx) < \infty$, and let T be the stopping time constructed in* (1.6)*. Then, for all $\beta \in [0, 1/4)$,*

$$\mathbb{E}_0 T^\beta < \infty. \tag{8.2}$$

The proof of Theorem 8.2 uses a result similar to Theorem 4 (ii) in [9] and Theorem 2 in [8], which is of independent interest and may also serve as another example for Theorem 5.1. We consider the 'clock'

$$U_r := \inf\big\{t > 0\colon \ell^0[0,t] + \ell^\nu[0,t] = r\big\}$$

and random measures ξ and η on the positive reals given by

$$\xi[0,r] := \ell^0[0,U_r], \qquad \eta[0,r] := \ell^\nu[0,U_r], \quad r \geq 0.$$

Proposition 8.3. *Let ξ and η be defined as above and let $S := \inf\{t > 0\colon \xi[0,t] = \eta[0,t]\}$. Then $\mathbb{E}_0 S^{1/2} = \infty$, but for some $c > 0$ we have $\mathbb{P}_0\{S > t\} \leq ct^{-1/2}$, for all $t \in \mathbb{R}$.*

Proof. The proof of $\mathbb{E}_0 S^{1/2} = \infty$ is very similar to Theorem 2 in [8] and is therefore omitted. We prove here the upper bound for the tail asymptotics (only this part is needed). This result is similar to Theorem 6 (ii) in [8], but due to the specific form of S we can use a more direct argument.

For any $i \in \mathbb{N}$ let $Y_i = \eta\{s \geq 0\colon i < \xi[0,s] \leq i+1\}$. Using the second Ray-Knight theorem and the assumption on the first moment of ν as in (8.1), we see that the sequence $Y_1, Y_2, \dots$ is an i.i.d. sequence of random variables with mean one and finite variance. Define, for $n \in \mathbb{N}$,

$$R_n := \sum_{i=1}^n 1 + Y_i, \qquad S_n := \sum_{i=1}^n 1 - Y_i.$$

Let $\sigma := \inf\{n \geq 1\colon S_n < 0\}$ and fix $a \in (0, 1/2)$. Then, for any $t > 0$,

$$\mathbb{P}_0\{S > t\} \leq \mathbb{P}_0\{R_\sigma > t\} \leq \mathbb{P}_0\{S_n \geq 0 \text{ for all } n \leq at\} + \mathbb{P}_0\{R_{\lfloor at \rfloor} > t\}.$$

By a classical result of Spitzer [22], see also [6, Theorem 1a in Section XII.7], the first term on the above right-hand side is bounded by a constant multiple of $(at)^{-1/2}$. By Chebyshev's inequality we have

$$\mathbb{P}_0\{R_{\lfloor at \rfloor} > t\} \leq \frac{1}{(t - 2\lfloor at \rfloor)^2}\, \mathbb{E}_0\big[(R_{\lfloor at \rfloor} - 2\lfloor at \rfloor)^2\big] = \frac{\lfloor at \rfloor}{(t - 2\lfloor at \rfloor)^2}\, \mathbb{E}_0[(1 - Y_1)^2],$$

which is bounded by a constant multiple of t^{-1}. This completes the proof. □

Proof of Theorem 8.2. The variable S, defined in Proposition 8.3, satisfies

$$S = \ell^0[0,T] + \ell^\nu[0,T] = 2\ell^0[0,T].$$

It remains to relate the tail behaviour of $\ell^0[0,T]$ (which we know) to that of T (which we require). To this end we observe that for $\theta \in \mathbb{R}$ and $t > \frac{3}{4}$, using [17, Theorem 6.10],

$$\mathbb{P}_0\Big\{ \inf_{s>t} \frac{1}{\sqrt{s/\log s}} \ell^0[0,s] < 1/\theta \Big\} = \mathbb{P}_0\Big\{ \inf_{s>t} \frac{1}{\sqrt{s/\log s}} \max_{0\le r\le s} |B_r| < 1/\theta \Big\}$$
$$\le \sum_{k=0}^{\infty} \mathbb{P}_0\Big\{ \frac{1}{\sqrt{t+k}} \max_{0\le r\le t+k} |B_r| < \frac{2}{\theta\sqrt{\log(t+k)}} \Big\}.$$

By a step in the proof of Chung's law of the iterated logarithm, see e.g. [10, (2.1)],

$$\mathbb{P}_0\Big\{ \frac{1}{\sqrt{t}} \max_{0\le r\le t} |B_r| < x \Big\} \le \frac{4}{\pi} e^{-\frac{\pi^2}{8x^2}}, \quad x > 0,$$

and hence we have

$$\mathbb{P}_0\Big\{ \inf_{s>t} \frac{1}{\sqrt{s/\log s}} \ell^0[0,s] < 1/\theta \Big\} \le t^{-1/4},$$

for a sufficiently large constant θ. For sufficiently large t we have

$$\mathbb{P}_0\Big\{ \frac{T}{\theta^2 \log T} > t \Big\} \le \mathbb{P}_0\{\ell^0[0,T] > \sqrt{t}\} + \mathbb{P}_0\Big\{ \inf_{s>t} \frac{1}{\sqrt{s/\log s}} \ell^0[0,s] < 1/\theta \Big\},$$

and the right hand side in this inequality is bounded by a constant multiple of $t^{-1/4}$. The result follows directly by integration. □

Next we turn to unbiased shifts T embedding a measure $\nu \neq \delta_0$, which need neither be stopping times, nor nonnegative. We conjecture that any such shift satisfies $\mathbb{E}_0|T|^{1/4} = \infty$. At the moment we can only prove the following weaker result.

Theorem 8.4. *If T is an unbiased shift embedding a probability measure $\nu \neq \delta_0$, then*

$$\mathbb{E}_0 \sqrt{|T|} = \infty.$$

Proof. The idea of this proof is due to Alex Cox. We work under the probability measure $\mathbb{P}_0$. By definition of an unbiased shift $B^+ := (B_{T+t} - B_T \colon t \ge 0)$ and $B^- := (B_{T-t} - B_T \colon t \ge 0)$ are independent Brownian motions. Moreover, the pair (B^+, B^-) is independent of B_T. Assume that $B_T \ge x$, where $x > 0$ is chosen such that $\nu[x,\infty) > 0$. (If there is no such $x > 0$ we find an $x < 0$ such that $\nu(-\infty, x] > 0$ and assume $B_T \le x$.) If $T > 0$, then $B^-_T = -B_T \le -x$, so that

$$T \ge S^- := \inf\{t \ge 0 \colon B^-_t = -x\}.$$

If $T < 0$, then $B^+_{-T} = -B^-_T \le -x$, so that

$$-T \ge S^+ := \inf\{t \ge 0 \colon B^+_t = -x\}.$$

Hence $|T| \ge S^- \wedge S^+ =: S$. It is well-known that $\mathbb{E}_0\sqrt{S^-} = \infty$ and $\mathbb{E}_0\sqrt{S^+} = \infty$. Since S^- and S^+ are independent, this property transfers to S. It follows that

$$\mathbb{E}_0 \sqrt{|T|} \ge \mathbb{E}_0 \mathbf{1}\{B_T \ge x\} \sqrt{S} = \nu[x,\infty)\, \mathbb{E}_0 \sqrt{S} = \infty.$$ □

Unbiased shifts embedding δ_0 also have bad moment properties if they are nonnegative (or, by time-reversal, nonpositive) but not identically zero. The result can be compared with Theorem 3 (i) in [8]. However, the proofs are very different.

Theorem 8.5. *If $T \geq 0$ is an unbiased shift such that $\mathbb{P}_0\{B_T = 0\} = 1$ and $\mathbb{P}_0\{T > 0\} > 0$, then*

$$\mathbb{E}_0 T = \infty.$$

Proof. We assume for contradiction that $m := \mathbb{E}_0 T < \infty$. Define a probability measure $\mathbb{P}^*$ on Ω by setting $\mathbb{E}_{\mathbb{P}^*} f(B) = \frac{1}{m}\mathbb{E}_0 \int_0^T f(\theta_s B)\, ds$ for each bounded nonnegative measurable function f. By Lemma 3.2, $\mathbb{P}^*$ is stationary. To show that, on the invariant σ-algebra $\mathcal{I}$, the process B has the same distribution under $\mathbb{P}^*$ as under $\mathbb{P}_0$, take $A \in \mathcal{I}$ and recall from Theorem 3.4 that $\mathbb{P}_0\{B \in A\} \in \{0, 1\}$. But $\mathbb{P}^*\{B \in A\} = \frac{1}{m}\mathbb{E}_0 \mathbf{1}\{B \in A\} T = 0$ or 1 according as $\mathbb{P}_0\{B \in A\} = 0$ or 1, as required. By [23, Theorem 2] we infer from this that

$$\frac{1}{t}\int_0^t \mathbb{P}_0\{\theta_s B \in \cdot\}\, ds \to \mathbb{P}^*\{B \in \cdot\}, \quad t \to \infty,$$

with respect to the total variation norm. On the other hand, for every $r > 0$,

$$\frac{1}{t}\int_0^t \mathbb{P}_0\{|B_s| \leq r\}\, ds \to 0, \quad t \to \infty,$$

implying $\mathbb{P}^*\{|B_0| \leq r\} = 0$ for all $r > 0$, which is a contradiction. □

In contrast to the two theorems above, we shall see below that unbiased shifts can have *good* moment properties if they can assume both signs.

Example 8.6. We construct a nonzero unbiased shift T embedding δ_0, which has $\mathbb{E}e^{\lambda|T|} < \infty$ for some $\lambda > 0$. Let $\{(a_i, b_i) \colon i \in \mathbb{Z}\}$ be the countable collection of maximal nonempty intervals (a, b) with the property that $B_t \neq 0$ for all $a < t < b$ and $|B_s| \geq 1$ for some $s \in (a, b)$. We assume that the collection is ordered such that $b_i < a_{i+1}$ for all $i \in \mathbb{Z}$. We define an allocation rule τ by the requirement that, for $b_i < s < a_{i+1}$,

$$\tau(s) = \begin{cases} \sup\{r < a_{i+1} \colon \ell^0(r, a_{i+1}) = \ell^0(b_i, s)\}, & \text{if } \ell^0(b_i, s) \leq \frac{1}{2}\ell^0(b_i, a_{i+1}), \\ \inf\{r > b_i \colon \ell^0(s, a_{i+1}) = \ell^0(b_i, r)\}, & \text{if } \ell^0(b_i, s) > \frac{1}{2}\ell^0(b_i, a_{i+1}). \end{cases}$$

It is easy to see that τ balances ℓ^0 with itself, and hence by Theorem 1.2, we have that $T = \tau(0)$ is an unbiased shift embedding δ_0. Moreover, we have $|T| \leq S_1 + S_2$ where $S_1 = \inf\{t > 0 \colon |B_t| = 1\}$ and $S_2 = -\sup\{t < 0 \colon |B_t| = 1\}$. S_1 and S_2 are obviously independent and identically distributed, and it is easy to see that they, and hence $|T|$, have the required moment property.

Remark 8.7. If $T \geq 0$ is an unbiased shift such that $\mathbb{P}_0\{B_T = 0\} = 1$ and $\mathbb{P}_0\{T > 0\} > 0$, then we conjecture that $\mathbb{E}_0\sqrt{T} = \infty$ (strengthening Theorem 8.5), but we cannot prove this without additional assumptions. One such assumption (covering T_r defined in (1.1) for $r > 0$) is that $\mathbb{P}_0\{T > s\} > 0$ for some $s > 0$ such that $\{T > s\}$ is $\mathbb{P}_0$-almost surely in the σ-algebra generated by $\{B_t \colon t \leq s\}$. Indeed, in this case we have

$$\mathbb{E}_0\sqrt{|T|} \geq \mathbb{E}_0\mathbf{1}\{T > s\}\sqrt{T} \geq \mathbb{E}_0\mathbf{1}\{T > s\}\sqrt{s + T_0 \circ \theta_s},$$

where $T_0 := \inf\{t > 0 \colon B_t = 0\}$. By the Markov property

$$\mathbb{E}_0 \sqrt{|T|} \geq \mathbb{E}_0 \mathbf{1}\{T > s\} \mathbb{E}_{B_s} \sqrt{T_0} = \infty,$$

since $\mathbb{E}_x \sqrt{T_0} = \infty$ for all $x \neq 0$ and $\mathbb{P}_0\{B_s = 0\} = 0$. Note that this argument does not use that T is unbiased.

9 Concluding remarks

In this section we make some comments on our results and their possible extensions. We begin with the following counterpart of Theorem 5.1 for simple point processes.

Theorem 9.1. *Let ξ and η be invariant simple point processes on $\mathbb{R}$ defined on some probability space equipped with a flow and an invariant σ-finite measure. Assume that ξ and η have finite intensities and that*

$$\mathbb{E}[\xi[0,1] \mid \mathcal{I}] = \mathbb{E}[\eta[0,1] \mid \mathcal{I}].$$

Then

$$\tau(s) := \inf\{t \geq s \colon \xi[s,t] = \eta[s,t]\}, \quad s \in \mathbb{R}, \tag{9.1}$$

is an allocation rule balancing ξ and η.

Theorem 9.1 is a one-sided (and one-dimensional) version of the stable matching procedure described in [8]. It can be proved by adapting the ideas of Theorem 5.1 to a discrete and therefore much simpler setup.

Remark 9.2. In the point process case the allocation rule (9.1) is right-stable in the sense of Section 7 and it is not difficult to show that it is the unique right-stable allocation balancing ξ and η. We conjecture that this uniqueness property also holds in the general case and therefore can be added to Theorem 7.1.

Next we comment on the extension of our results to other processes with stationary increments.

Remark 9.3. Let $X = (X_t)_{t\in\mathbb{R}}$ be a right-continuous real-valued stochastic process with left-hand limits and $X_0 = 0$, defined on its canonical probability space $(\Omega, \mathcal{A}, \mathbb{P}_0)$. Assume that X has stationary increments. Defining $\mathbb{P}_x$, $x \in \mathbb{R}$, as before, the definition (1.2) still yields a stationary measure. Assume that there is a (measurable) family ℓ^x, $x \in \mathbb{R}$, of local times satisfying (2.4). Under technical assumptions on this family (weaker than (2.5) and (2.6)) Theorem 1.2 still holds. One does not need to assume that the local times are diffuse. However, one cannot expect the existence results (e.g. Theorem 1.4) to hold in such a general setting.

Remark 9.4. Consider the setting of Example 9.3 and assume in addition that X is a Lévy process. There are assumptions on X that guarantee the existence of diffuse local time measures that are perfect in the sense of (2.5) and (2.6), see [1]. Under these assumptions Theorems 1.3, 1.4, 1.1, 3.4, and 3.3 all hold. Also the minimality properties stated in Theorems 7.4 and 7.5 as well as the stability assertion of Theorem 7.1 remain true in this more general setting.

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References

[1] Barlow, M.T., Perkins, E.A. and Taylor, S.J. (1986). Two uniform intrinsic constructions for the local time of a class of Lévy processes. *Illinois J. of Math.* **30**, 19–65.

[2] Barlow, M.T. and Yor, M. (1981). (Semi-) martingale inequalities and local times. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **55**, 237–254.

[3] Bertoin, J. and Le Jan, Y. (1992). Representation of measures by balayage from a regular recurrent point. *Ann. Probab.* **20**, 538–548.

[4] Cox, A. M. G. and Hobson, D. G. (2006). Skorokhod embeddings, minimality and non-centred target distributions. *Probab. Theory Related Fields* **135**, 395–414.

[5] Dudley, R.M. (2003). *Real Analysis and Probability.* Cambridge University Press, Cambridge.

[6] Feller, W. (1966). *An Introduction to Probability Theory and its Applications. Volume II.* Second edition. Wiley, New York.

[7] Geman D. and Horowitz J. (1973). Occupation times for smooth stationary processes. *Ann. Probab.* **1**, 131–137.

[8] Holroyd, A.E., Pemantle, R., Peres, Y. and Schramm, O. (2009). Poisson Matching. *Annales de l'institut Henri Poincaré (B)* **45**, 266–287.

[9] Holroyd, A.E. and Peres, Y. (2005). Extra heads and invariant allocations. *Ann. Probab.* **33**, 31–52.

[10] Jain, N. C. and Pruitt, W.E. (1975). The other law of the iterated logarithm. *Ann. Probab.* **3**, 1046–1049.

[11] Kagan, Y.A. and Vere-Jones, D. (1996). Problems in the modelling and statistical analysis of earthquakes. In Athens Conference on Applied Probability and Time Series, Vol 1: Applied Probability (Springer Lecture Notes in Statistics 114), eds C.C. Heyde, Y..V. Prohorov, R. Pyke, S.T. Rachev, Springer, Berlin, 398–425.

[12] Kallenberg, O. (2002). *Foundations of Modern Probability.* Second Edition, Springer, New York.

[13] Last, G. (2010). Modern random measures: Palm theory and related models. *New Perspectives in Stochastic Geometry.* (W. Kendall und I. Molchanov, eds.). Oxford University Press.

[14] Last, G. and Thorisson, H. (2009). Invariant transports of stationary random measures and mass-stationarity. *Ann. Probab.* **37**, 790–813.

[15] Mandelbrot, B. (1982). *The Fractal Geometry of Nature*, Freeman and Co., San Francisco.

[16] Mecke, J. (1967). Stationäre zufällige Maße auf lokalkompakten Abelschen Gruppen. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **9**, 36–58.

[17] Mörters, P. and Peres, Y. (2010). *Brownian Motion.* Cambridge University Press, Cambridge.

[18] Monroe, I. (1972). On embedding right continuous martingales in Brownian motion. *Ann. Math. Statist.* **43**, 1293–1311.

[19] Obłój, J.K. (2004). The Skorokhod embedding problem and its offspring. *Probability Surveys* **1**, 321–392.

[20] Perkins, E. (1981). A global intrinsic characterisation of local time. *Ann. Probab.* **9**, 800–817.

[21] Revuz, D., Yor, M. (1999). Continuous Martingales and Brownian Motion. Springer, Berlin.

[22] Spitzer, F.L. (1960). A Tauberian theorem and its probability interpretation. *Trans. Amer. Math. Soc.* **94**, 150–169.

[23] Thorisson, H. (1996). Transforming random elements and shifting random fields. *Ann. Probab.* **24**, 2057–2064.

[24] Thorisson, H. (2000). *Coupling, Stationarity, and Regeneration.* Springer, New York.

[25] Zähle, U. (1991). Self-similar random measures III. Self-similar random processes. *Math. Nachr.* **151**, 121–148.